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A Note on Snevily's Conjecture in Set Systems

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Abstract

Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a set of s nonnegative integers and $K = \{k_1, k_2, \ldots, k_r\}$ be a set of integers satisfying $\max l_j < \min k_i$. Let $\mathcal{F}$ be an $\mathcal{L}$-intersecting family of subsets of $\{1, 2, \ldots, n\}$ such that $|F| \in K$ for every $F \in \mathcal{F}$. Let $\{0, 1, \ldots, s-r-1\} \subseteq \mathcal{L}$ with $1 < r < s$ and K be any set of integers with $\min k_i > s - r$ and $K \cap \mathcal{L} = \emptyset$. Then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$. These results confirm Snevily's conjecture on set systems partially.

Keywords: Ray-Chaudhuri-Wilson Theorem, Snevily's conjecture

AMS Classifications: 05D05.

1 Introduction

Throughout this paper X will denote the set $[n] = \{1, 2, \ldots, n\}$, $K = \{k_1, k_2, \ldots, k_r\}$ and $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ will be two sets of nonnegative integers with $\max l_j < \min k_i$, and $\mathcal{F}$ will denote a family of subsets of X such that $|E \cap F| \in \mathcal{L}$ for every pair of distinct subsets $E, F \in \mathcal{F}$ and $|F| \in K$ for every $F \in \mathcal{F}$. In this paper, we are interested in the following conjectures of Snevily, which give some upper bounds on the size of $\mathcal{F}$.

Conjecture 1.1 (Snevily [6]) *For any K and $\mathcal{L}$ with $\max l_j < \min k_i$, $|\mathcal{F}| \leq \binom{n}{s}$.*

Conjecture 1.2 (Snevily [6]) *For any K and $\mathcal{L}$ with $\max l_j < \min k_i$, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.*

Note that Conjecture 1.2 is weaker than Conjecture 1.1. We will first present some results related to this problem that have been obtained by others.

Theorem 1.3 (Ray-Chaudhuri and Wilson [2]) *If* $K = \{k\}$ *and* $\mathcal{L}$ *is any set of nonnegative integers with* $\max l_j < k$*, then* $|\mathcal{F}| \le \binom{n}{s}$.

Theorem 1.4 (Snevily [5]) *If* K *and* $\mathcal{L}$ *are any sets such that* $\max l_j < \min k_i$*, then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

Theorem 1.5 (Snevily [6]) *Let* K *and* $\mathcal{L}$ *be sets of nonnegative integers such that* $\max l_j < \min k_i$*. Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

Theorem 1.6 (Snevily [6]) *Conjecture* 1.1 *holds for* $\mathcal{L} = \{0, 1, \ldots, s-1\}$.

Theorem 1.7 (Hwang and Sheikh [4]) *Conjecture* 1.2 *holds when* K *is a consecutive set.*

Theorem 1.8 (Chen and Liu [3]) *Conjecture* 1.2 *holds for* $\mathcal{L} = \{1, 2, \ldots, s\}$.

Clearly, Ray-Chaudhuri-Wilson Theorem (Theorem 1.3) implies that Conjecture 1.1 holds for $r = 1$ and Theorem 1.4 implies that Conjecture 1.2 holds for $r \ge s$. Here, we will prove Conjecture 1.2 holds for $\{0, 1, \ldots, s-r-1\} \subseteq \mathcal{L}$ with $1 < r < s$.

Theorem 1.9 *Let* $\{0, 1, \ldots, s-r-1\} \subseteq \mathcal{L}$ *with* $1 < r < s$ *and* K *be any set of integers with* $\min k_i > s - r$ *and* $K \cap \mathcal{L} = \emptyset$*. Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

2 Proof of Theorem 1.9

In this section, we give a proof for Theorem 1.9 using the techniques in [1, 3, 4, 5, 6].

Proof of Theorem 1.9 Let $\{0, 1, \ldots, s-r-1\} \subseteq \mathcal{L}$ with $1 < r < s$ and K be any set of integers with $\min k_i > s - r$ and $K \cap \mathcal{L} = \emptyset$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$. With each set $F_i \in \mathcal{F}$, we associate its characteristic vector $v_i = (v_{i_1}, \ldots, v_{i_n}) \in \mathbb{R}^n$, where $v_{i_j} = 1$ if $j \in F_i$ and $v_{i_j} = 0$ otherwise.

Recall that a polynomial in n variables is multilinear if its degree in each variable is at most 1. Let us restrict the domain of the polynomials we will work with to the n-cube $\Omega = \{0,1\}^n \subseteq \mathbb{R}^n$. Since in this domain ${x_i}^2 = x_i$ for each variable, every polynomial in our proof is multilinear.

For each $F_i \in \mathcal{F}$, define

$$f_i(x) = \prod_{j=1}^{s} (v_i \cdot x - l_j).$$

Then $f_i(v_i) \neq 0$ for every $1 \leq i \leq m$ and $f_i(v_j) = 0$ for $i \neq j$.

Let $\mathcal{G} = \{G_1, \ldots, G_p\}$ be the family of subsets of $X = [n] \setminus \{n\}$ with size at most $s-1$, which is ordered by size, that is, $|G_i| \leq |G_j|$ if $i < j$, where $p = \sum_{i=0}^{s-1} \binom{n-1}{i}$. Let u_i denote the characteristic vector of G_i. For $i = 1, \ldots, p$, we define

$$g_i(x) = (1 - x_n) \prod_{j \in G_i} x_j.$$

Since $g_i(u_i) \neq 0$ for every $1 \leq i \leq p$ and $g_i(u_j) = 0$ for any $j < i$, $\{g_i(x) | 1 \leq i \leq p\}$ is a linearly independent family.

Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be the family of subsets of $[n]$ with size at most $s - r$ which contain n, where $q = \sum_{i=0}^{s-r-1} \binom{n-1}{i}$. We order the members of $\mathcal{H}$ such that $|H_i| \leq |H_j|$ if $i < j$. Let w_i be the characteristic vector of H_i. For $i = 1, \ldots, q$, define

$$h_i(x) = \left(\prod_{l=1}^{r} (\sum_{j=1}^{n} x_j - k_l) \right) (\prod_{j \in H_i} x_j)$$

Note that $h_i(w_j) = 0$ for any $j < i$ and $h_i(w_i) \neq 0$ for every $1 \leq i \leq q$ since $\min k_i > s - r$, and thus $\{h_i(x) | 1 \leq i \leq q\}$ is a linearly independent family.

We will show that the polynomials in

$$\{f_i(x) | 1 \leq i \leq m\} \cup \{g_i(x) | 1 \leq i \leq p\} \cup \{h_i(x) | 1 \leq i \leq q\}$$

are linearly independent. Suppose that we have a linear combination of these polynomials that equals zero:

$$\sum_{i=1}^{m} \alpha_i f_i(x) + \sum_{i=1}^{p} \beta_i g_i(x) + \sum_{i=1}^{q} \gamma_i h_i(x) = 0. \tag{2.1}$$

We will prove that the coefficients must be zero. First by substituting the characteristic vector v_i of F_i with $n \in F_i$ into equation (2.1), we get $\alpha_i f_i(v_i) = 0$. Since $f_i(v_i) \neq 0$, we have $\alpha_i = 0$ if $n \in F_i$. It follows that

$$\sum_{n \notin F_i} \alpha_i f_i(x) + \sum_{i=1}^{p} \beta_i g_i(x) + \sum_{i=1}^{q} \gamma_i h_i(x) = 0. \tag{2.2}$$

Then we substitute the characteristic vector w_i of H_i into the equation (2.2) in order of non-decreasing size of H_i with $1 \leq i \leq q$. Since $n \in H_i$, $g_j(w_i) = 0$ for every $1 \leq j \leq p$. For each F_j with $n \notin F_j$, we have $H_i \nsubseteq F_j$. Since $|H_i| \leq s - r$ and $\{0, 1, \ldots, s-r-1\} \subseteq \mathcal{L}$, we have $|F_j \cap H_i| \leq s - r - 1$ and so $|F_j \cap H_i| \in \mathcal{L}$. Thus, $f_j(w_i) = 0$ for each F_j with $n \notin F_j$. Note that $h_j(w_i) = 0$ for any $i < j$ and $h_i(w_i) \neq 0$ for every $1 \leq i \leq q$ since $\min k_i > s - r$, it is easy to obtain that $\gamma_i h_i(w_i) = 0$ when evaluating equation (2.2) with $x = w_i$. We get $\gamma_i = 0$ for $1 \leq i \leq q$. Thus equation (2.2) reduces to

$$\sum_{n \notin F_i} \alpha_i f_i(x) + \sum_{i=1}^{p} \beta_i g_i(x) = 0. \tag{2.3}$$

Let $F_i^* = F_i \cup \{n\}$ if $n \notin F_i$. We substitute the characteristic vector v_i^* of F_i^* into the equation (2.3). Note that $f_j(v_i^*) = f_j(v_i)$ for each j with $n \notin F_j$ and $g_j(v_i^*) = 0$ for $1 \leq j \leq p$. We get $\alpha_i f_i(v_i^*) = \alpha_i f_i(v_i) = 0$ which implies $\alpha_i = 0$ if $n \notin F_i$. It is left to show that $\gamma_i = 0$ for $1 \leq i \leq q$. Since the family $\{g_i(x) | 1 \leq i \leq p\}$ is linearly independent, we are done.

To complete the proof, simply note that each polynomial in $\{f_i(x) | 1 \leq i \leq m\} \cup \{g_i(x) | 1 \leq i \leq p\} \cup \{h_i(x) | 1 \leq i \leq q\}$ can be written as a linear combination of the multilinear polynomials of degree at most s. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s} \binom{n}{i}$. It follows that

$$m + p + q = |\mathcal{F}| + \sum_{i=0}^{s-1} \binom{n-1}{i} + \sum_{i=0}^{s-r-1} \binom{n-1}{i} \leq \sum_{i=0}^{s} \binom{n}{i}$$

which implies

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}.$$

This completes the proof of the theorem. □

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